Mathematics: analysis and approaches
Higher level
Paper 3

Name

Date: ____________________

1 hour 15 minutes

Instructions to candidates

- Write your name in the box above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

exam: 3 pages

Answer all questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 30]

This question analyses different aspects of rational functions involving the sine and cosine functions.

Consider the function $y = \dfrac{1}{\sin x + \cos x}$.

(a) (i) The domain of the function is defined as $0 \le x \le 2\pi, x \ne c, x \ne d$ where $c < d$. Write down the exact value of c and the exact value of d. [2]

(ii) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sin x - \cos x}{(\sin x + \cos x)^2}$. [3]

(iii) The graph of the function has a local minimum at point S and a local maximum at point T. Find the exact coordinates of S and the exact coordinates of T. [4]

(iv) Sketch a graph of the function labelling all asymptotes. [4]

Consider the function $y = \dfrac{a + b\sin x}{b + a\sin x}$ defined for $0 \le x \le 2\pi$, where $a, b \in \mathbb{R}$ and $0 < a < b$.

(b) (i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(b^2 - a^2)\cos x}{(b + a\sin x)^2}$. [3]

(ii) The graph of y has a horizontal tangent at point M and point N. Their x-coordinates are x_m and x_n. Given $x_m < x_n$, find the exact coordinates of M and of N. [4]

Consider the function $f(x) = \dfrac{a + b\sin x}{b + a\sin x}$ defined for $x \in \mathbb{R}$, where $a, b \in \mathbb{R}$ and $0 < a < b$.

(c) (i) Explain why the graph of f cannot have a vertical asymptote. [2]

(ii) Deduce a formula for the values of x where f has a horizontal tangent. Express the formula in terms of n where n is a positive integer. [2]

Consider the function $g(x) = \dfrac{3 + 4\sin x}{4 + 3\sin x}$ defined for $0 \le x \le 2\pi$.

(d) (i) Sketch the graph of g indicating the coordinates of both x-intercepts. [3]

(ii) Find the total area enclosed by the graph of g, the y-axis and the x-axis. [3]

2. [Maximum mark: 25]

This question involves modelling the motion of two points in a three-dimensional coordinate system consisting of the mutually perpendicular axes x, y and z. The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors in the positive direction of the x -axis, y -axis and z -axis, respectively. For this question, the unit vectors have a length of 1 cm.

The point P is moving such that its position vector at a time of t seconds is given by

$$\vec{\mathrm{OP}} = (1+t)\boldsymbol{i} + (2-2t)\boldsymbol{j} + (3t-1)\boldsymbol{k} \text{ where } t \geq 0.$$

(a) Find the coordinates of P when $t = 0$ seconds. [2]

(b) Show that P moves along the line L with Cartesian equations

$$x - 1 = \frac{y-2}{-2} = \frac{z+1}{3}.$$ [2]

The plane $\prod$ has Cartesian $2x + y + z = 6$.

(c) (i) P lies on the plane $\prod$ when $t = m$ seconds. Find the value of m. [2]

(ii) State the coordinates of P when it lies on plane $\prod$. [2]

(iii) Find the exact distance P travels from when $t = 0$ to when it meets plane $\prod$. [3]

The position vector of another point, Q, at a time of t seconds is given by

$$\vec{\mathrm{OQ}} = \begin{pmatrix} t^2 \\ 1-t \\ 1-t^2 \end{pmatrix} \text{ where } t \geq 0.$$

(d) (i) Find the time, t, at which the distance from Q to the origin is a minimum. [5]

(ii) Find the coordinates of Q at this time. [2]

Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the position vectors of Q at times $t = 0$, $t = 1$ and $t = 2$ respectively.

(e) (i) Show that the equation $\boldsymbol{a} - \boldsymbol{b} = \lambda(\boldsymbol{b} - \boldsymbol{c})$ has no solution for λ. [5]

(ii) Hence, deduce that the path of Q is not a straight line. [2]